JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 1, January–February 1994

Calculus of Variations Derivation of the Minimax Linear-Quadratic (H_∞) Controller

Raymond A. Mills* and Arthur E. Bryson†
Stanford University, Stanford, California 94305

The *linear-quadratic* best controller for the *worst* bounded disturbances (*LQW* controller), or so-called H_∞ controller, is derived as a differential game between the controller and disturber using the calculus of variations. This derivation explicitly shows that for the full-order LQW controller the worst measurement disturbance is zero (!) and the controller initial conditions must be set equal to the plant initial conditions for a well-posed differential game. As in previous derivations, the worst process disturbance is shown to be a feedback on the plant states. The derivation yields necessary conditions for reduced-order and higher order LQW controllers, useful in multiple-plant optimization for robustness. A helicopter near hover is used to illustrate differences between LQW and linear-quadratic-Gaussian (LQG) control. This comparison suggests the relative merits of LQG and LQW control design and shows that a special case of LQW control called infinite-disturbance LQW ("optimal" H_∞) control is not practical.

Introduction

OVER the past decade, "H_∞ control" has been developed and popularized.[1–5] It denotes minimizing the H_∞ norm of the closed-loop transfer function from disturbances to outputs and controls. Although theoretical work continues, some methods are being proposed for practical use.[6–8]

This paper presents H_∞ control as the limit of a linear-quadratic differential game between the controller and the disturber, which we call the *LQW* controller. The motivation for this derivation comes from the full-state feedback derivation of Ref. 9. Many of its interpretations extend to the compensation case. Reference 10 formulates classes of LQW control analysis and design problems. Reference 11 combines techniques from Ref. 9 with worst bounded plant parameter changes from nominal to produce LQW state feedback robust to plant parameter changes.

We are not the first to derive the LQW (H_∞) compensator from a differential game using the calculus of variations. However, the derivation here differs from previous ones in the literature[12,13] in that we assume a controller form a priori. The advantage of the approach in Refs. 12 and 13 is that the full-order, linear controller, without direct feedthrough from measurement to control, is shown to be the optimal controller. The advantage of our approach is a simpler derivation that yields necessary conditions for reduced-order LQW controllers and higher order LQW controllers; these are useful for reduced-order and multiple-plant optimal controller design using gradient techniques.[14,15] Multiple-plant controller optimization is used to increase robustness to plant parameter changes.[16,17]

This paper contains an example of LQW control design for a helicopter near hover. The purpose of this example is to illustrate features of LQW control and differences between LQW and linear-quadratic-Gaussian (LQG) control that are of practical importance. Comparison to LQG control is interesting because LQG control is used in practice, and it is a special case of LQW control.[3,4] This example does not, of course, constitute a proof of the general relative merits of LQG and LQW control, but it does suggest them. We believe the qualitative statements made for this example apply in general based on experience and physical reasoning.

Received Sept. 12, 1992; revision received March 5, 1993; accepted for publication March 10, 1993.

*Ph.D. recipient, Department of Aeronautics and Astronautics; currently Senior Systems Engineer, AeroAstro Corporation, 520 Huntmar Park Drive, Herndon, VA 22070. Senior Member AIAA.

†Pigott Professor of Engineering, Department of Aeronautics and Astronautics, Durand Building. Honorary Fellow AIAA.

Equivalent H_∞/LQW Problem Statements

The H_∞ compensator problem[3,4] is to find a controller such that

$$\|G\|_\infty < \gamma \tag{1}$$

where, with the control loop closed,

$$z = Gd \tag{2}$$

and

$$z \triangleq \begin{bmatrix} \sqrt{Q}y \\ \sqrt{R}u \end{bmatrix} \tag{3}$$

$$d \triangleq \begin{bmatrix} \sqrt{R_w}w \\ \sqrt{R_v}v \end{bmatrix} \tag{4}$$

where y is the plant output, u is the control, w is the process disturbance, and v is the measurement disturbance. Thus, G is the *closed-loop* transfer matrix from the weighted process and measurement disturbances to the weighted outputs and controls. The weighting matrices Q, R, R_w, and R_v are symmetric positive-definite. The reason for the matrix square root on them will become apparent later. The weighting matrices are embedded transparently in G in Refs. 3 and 4 but are included explicitly here for practical reasons and for comparison with the standard LQG controller problem.

Four interpretations of the infinity norm of a transfer matrix G, $\|G\|_\infty$, are given in Ref. 18: in summary, the infinity norm measures 1) the peak gain from root-mean-square (rms) input to rms output, 2) the peak total energy gain from input to output, 3) the peak of the maximum singular value of $G(j\omega)$ over all ω, and 4) the peak steady-state response to a unit-amplitude sinusoid over all frequencies. From these interpretations in Ref. 18, it is clear that making $\|G\|_\infty$ small results in "small" outputs and controls in the presence of process and measurement disturbances. Thus, the objective of the H_∞ problem is to find the controller that produces a closed-loop transfer matrix with a small infinity norm, namely, $\|G\|_\infty < \gamma$ for a specified γ. The solution is derived in Ref. 3 and given and explained in Ref. 4.

Following the interpretation of the state-feedback H_∞ problem as a differential game in Ref. 9, a minimax compensator problem statement equivalent to Eq. (1) is

$$J_S = \min_{\mathcal{C}} \max_{\mathcal{D}} \int_0^\infty (y^TQy + u^TRu)\, dt \tag{5}$$

where the subscript S denotes "saddle-point" and

$$\mathcal{C} \triangleq \{x_{c_0}, A_c, B_c, C_c\} \tag{6}$$

$$\mathcal{D} \triangleq \{x_0, w, v\} \tag{7}$$

subject to

$$\dot{x} = Ax + Bu + \Gamma w, \qquad x(0) = x_0 \tag{8}$$

$$y = Cx \tag{9}$$

$$y_s = C_s x + v \tag{10}$$

$$\dot{x}_c = A_c x_c + B_c y_s, \qquad x_c(0) = x_{c_0} \tag{11}$$

$$u = C_c x_c \tag{12}$$

$$2 = x_0^T x_0 + x_{c_0}^T x_{c_0} \tag{13}$$

$$W = \int_0^\infty (w^T R_w w + v^T R_v v)\, dt \tag{14}$$

where x is the plant state, y_s is the measurement, x_c is the controller state, and W is an integral-norm constraint on the disturbance energy. This is a differential game between the controller, which minimizes the performance index in Eq. (5), and the disturber, which maximizes the performance index. This formulation implicitly assumes that the disturber knows both x and x_c, whereas the controller only knows x_c. The disturbances must be constrained or else they will be infinite, making the performance index infinite. It will be seen that if the controller initial condition is set to zero, the problem is ill-posed and that a well-posedness condition determines x_{c_0}.

Since a primary purpose of this paper is to interpret and compare, feedthrough terms [for example, an additional term "Du" on the right-hand side of Eq. (9)] are not considered because they complicate the algebra considerably while adding little insight. Results with feedthrough can be obtained by following the steps indicated here with the more complex feedthrough algebra. Similarly, the distribution matrix from v to y_s in Eq. (10) is set to identity for simplicity and because this is often the case in practice. We present the infinite-time case for simplicity; the finite-time case follows immediately with the inclusion of non-zero time derivatives.

The problem of Eqs. (5–14) resembles the minimax problem for the best controller for the worst bounded plant parameter changes from nominal for time-invariant plants.[16,17] It is different in that the disturbances w and v are functions of time, whereas the plant parameters are not. The occurrence of near-worst disturbances *at all times* is less probable than the occurrence of near-worst plant parameter changes. This suggests that the solution to the problem of Eqs. (5–14) may result in overly conservative controllers.

In deriving the solution to Eqs. (5–14) in the next section, it is shown that the best controls and worst process disturbances are feedbacks on the plant states. For the full-order controller, the worst measurement disturbances are zero because the initial controller states must equal the initial plant states for well-posedness. The disturber can increase J_S more by putting all of its energy into the process disturbances! As the disturbance energy $W \to \infty$, the control energy, and hence J_S, also $\to \infty$, but the ratio J_S/W tends to the finite value γ^2. For $J_S/W > \gamma^2$ a finite-disturbance LQW [so-called suboptimal H_∞ (Refs. 3 and 4)] solution results. The infinite-disturbance LQW (optimal H_∞) controller is very conservative with regard to deterministic disturbances but is infinitely sensitive to disturbance *noise* because of its infinite closed-loop bandwidth (Ref. 9 and later).

Verification that Eqs. (5–14) are equivalent to Eq. (1) with Eq. (2) is given in the next section by solving Eqs. (5–14) using the calculus of variations and obtaining the same controller solution as in Refs. 3 and 4. The problem statement of Eqs. (5–14) is suggested from Eq. (1) as follows, where now the initial conditions in $\mathcal{C}$ [Eq. (6)] and $\mathcal{D}$ [Eq. (7)] have no importance.

Write

$$\gamma = \min_{\mathcal{C}} \|G\|_\infty \tag{15}$$

Using Eq. (4) and writing the infinity norm in terms of the $\mathcal{L}_2$ norm, $\|\cdot\|_2$ (Ref. 18), Eq. (15) becomes

$$\gamma^2 = \min_{\mathcal{C}} \frac{\max_{\mathcal{D}\,(d\neq 0)} \|Gd\|_2^2}{\|d\|_2^2} \tag{16}$$

$$= \min_{\mathcal{C}} \frac{\max_{\mathcal{D}\,(d\neq 0)} \int_{-\infty}^{\infty} (d^TG^TGd)\, dt}{\int_{-\infty}^{\infty} (d^Td)\, dt} \tag{17}$$

where the definition of the $\mathcal{L}_2$ norm[18] has been used to go from Eq. (16) to Eq. (17). Using Eqs. (2–4) in Eq. (17), and letting all signals be zero for $t < 0$,

$$\gamma^2 = \frac{\min_{\mathcal{C}} \max_{\mathcal{D}\,(d\neq 0)} \int_0^\infty z^Tz\, dt}{\int_0^\infty (w^TR_w w + v^TR_v v)\, dt} \tag{18}$$

$$= \frac{J_S}{W} \tag{19}$$

Calculus of Variations LQW (H_∞) Controller Derivation

The purpose of this section is to derive necessary conditions for LQW controllers of any order. These necessary conditions lead to a closed-form solution for full-order LQW controllers. They are also useful for finding LQW controllers numerically when the controller order does not equal the plant order.

The derivation of the solution to Eqs. (5–14) is suggested by the LQG derivation in Ref. 17. Consider the entire dynamic system with state vector

$$x_a \triangleq \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{20}$$

and define

$$w_a \triangleq \begin{bmatrix} w \\ v \end{bmatrix} \tag{21}$$

yielding

$$\dot{x}_a = A_a x_a + \Gamma_a w_a \tag{22}$$

where

$$A_a = \begin{bmatrix} A & BC_c \\ B_cC_s & A_c \end{bmatrix} \tag{23}$$

$$\Gamma_a = \begin{bmatrix} \Gamma & 0 \\ 0 & B_c \end{bmatrix} \tag{24}$$

From Eqs. (8) and (11), the augmented-state initial condition is

$$x_{a_0} \triangleq x_a(0) = \begin{bmatrix} x_0 \\ x_{c_0} \end{bmatrix} \tag{25}$$

which with Eq. (13) allows Eq. (13) to be replaced by

$$2 = x_{a_0}^T x_{a_0} \tag{26}$$

The performance index in Eq. (5) can be written

$$J = \frac{1}{2}\int_0^\infty x_a^T Q_a x_a \, dt \tag{27}$$

where

$$Q_a = \begin{bmatrix} C^TQC & 0 \\ 0 & C_c^T R C_c \end{bmatrix} \tag{28}$$

Augment the performance index with the constraint equations (22), (14), and (26), using Lagrange multipliers λ, $\nu/2$, and $\eta/2$, respectively:

$$\bar{J} = \int_0^\infty \left[\frac{1}{2} x_a^T Q_a x_a + \lambda^T (A_a x_a + \Gamma_a w_a - \dot{x}_a)\right] dt - \frac{\nu}{2} \times \left[\int_0^\infty (w^T R_w w + v^T R_v v)\, dt - W\right] - \frac{\eta}{2}(x_{a_0}^T x_{a_0} - 2) \tag{29}$$

$$= \int_0^\infty \left[\frac{1}{2} x_a^T Q_a x_a + \lambda^T(A_a x_a + \Gamma_a w_a - \dot{x}_a) + \frac{1}{2} w_a^T R_a w_a\right] dt + \frac{\nu}{2} W - \frac{\eta}{2}(x_{a_0}^T x_{a_0} - 2) \tag{30}$$

where

$$R_a = -\nu \begin{bmatrix} R_w & 0 \\ 0 & R_v \end{bmatrix} \tag{31}$$

Now take the differential of $\bar{J}$ by varying x_a and w_a and taking the differentials of the controller state matrices and x_{a_0}:

$$d\bar{J} = \int_0^\infty (x_a^T Q_a + \lambda^T A_a)\delta x_a + (w_a^T R_a + \lambda^T \Gamma_a)\delta w_a - \lambda^T \delta\dot{x}_a + x_a^T \overline{dQ_a} x_a + \lambda^T (dA_a x_a + d\Gamma_a w_a)\, dt - \eta x_{a_0}^T dx_{a_0} \tag{32}$$

where

$$\overline{dQ_a} = \begin{bmatrix} 0 & 0 \\ 0 & C_c^T R \, dC_c \end{bmatrix} \tag{33}$$

$$dA_a = \begin{bmatrix} 0 & B\, dC_c \\ dB_c C_s & dA_c \end{bmatrix} \tag{34}$$

$$d\Gamma_a = \begin{bmatrix} 0 & 0 \\ 0 & dB_c \end{bmatrix} \tag{35}$$

Integrating by parts,

$$d\bar{J} = -\lambda^T \delta x_a |_0^\infty + \textstyle\int_0^\infty \big[(x_a^T Q_a + \lambda^T A_a + \dot{\lambda}^T)\delta x_a + (w_a^T R_a + \lambda^T \Gamma_a)\delta w_a + x_a^T \overline{dQ_a} x_a + \lambda^T (dA_a x_a + d\Gamma_a w_a)\big] \times dt - \eta x_{a_0}^T dx_{a_0} \tag{36}$$

For a stationary point, setting the coefficient of δx_a to zero,

$$\dot{\lambda}^T = -\lambda^T A_a - x_a^T Q_a, \qquad \lambda^T(\infty) = 0 \tag{37}$$

Setting the coefficient of δw_a to zero,

$$w_a = -R_a^{-1} \Gamma_a^T \lambda \tag{38}$$

Let

$$\lambda = \Lambda_a x_a, \qquad \Lambda_a(\infty) = 0 \tag{39}$$

so

$$\dot{\lambda} = \dot{\Lambda}_a x_a + \Lambda_a \dot{x}_a \tag{40}$$

Then using the transpose of Eqs. (37) and (22) with Eq. (38) in Eq. (40),

$$-A_a^T \Lambda_a x_a - Q_a x_a = \dot{\Lambda}_a x_a + \Lambda_a (A_a x_a - \Gamma_a R_a^{-1} \Gamma_a^T \Lambda_a x_a) \tag{41}$$

Balancing coefficients of x_a and realizing $\dot{\Lambda}_a$ is zero in steady state, assuming a stable closed-loop system,

$$0 = A_a^T \Lambda_a + \Lambda_a A_a - \Lambda_a \Gamma_a R_a^{-1} \Gamma_a^T \Lambda_a + Q_a \tag{42}$$

a Riccati equation for Λ_a dependent on the as yet unknown controller state matrices.

From Eqs. (36–39), now

$$d\bar{J} = \mathrm{tr}(P_a \, dZ_a) + x_{a_0}^T (\Lambda_a - \eta I)\, dx_{a_0} \tag{43}$$

where

$$P_a = \int_0^\infty x_a x_a^T \, dt \tag{44}$$

$$dZ_a = \overline{dQ_a} + \Lambda_a (dA_a - d\Gamma_a R_a^{-1} \Gamma_a^T \Lambda_a) \tag{45}$$

and the identity $x^T A x \equiv \mathrm{tr}(xx^T A)$ has been used.

Define

$$P_a(t) \triangleq \int_0^t x_a x_a^T \, d\tau \tag{46}$$

Then

$$\dot{P}_a(t) = X_a \tag{47}$$

where

$$X_a = x_a x_a^T \tag{48}$$

Using Eqs. (22) and (48) in the time derivative of Eq. (48),

$$\dot{X}_a = A_P X_a + X_a A_P \tag{49}$$

where

$$A_P = A_a - \Gamma_a R_a^{-1} \Gamma_a^T \Lambda_a \tag{50}$$

Integrating both sides from 0 to t and using Eq. (47),

$$X_a(t) - X_a(0) = A_P P_a(t) + P_a(t) A_P^T \tag{51}$$

Then realizing that $\dot{P}_a(t) = X_a$ will be zero in steady state for a stable A_P and using Eq. (48),

$$0 = A_P P_a + P_a A_P^T + x_{a_0} x_{a_0}^T \tag{52}$$

a Lyapunov equation for $P_a = P_a^T$.

To expand $P_a \, dZ_a$, realizing that only the diagonal blocks are needed due to the trace operator, partition Λ_a and P_a:

$$\Lambda_a \triangleq \begin{bmatrix} \Lambda & \Lambda_2 \\ \Lambda_2^T & \Lambda_c \end{bmatrix} \tag{53}$$

$$P_a \triangleq \begin{bmatrix} P & P_2 \\ P_2^T & P_c \end{bmatrix} \tag{54}$$

Then

$$\operatorname{tr}(P_a\,\mathrm{d}Z_a) = \operatorname{tr}\left[(P\Lambda_2 + P_2\Lambda_c)\,\mathrm{d}B_c\left(C_s + \frac{1}{\nu}R_v^{-1}B_c^T\Lambda_2^T\right)\right]$$

$$+ \operatorname{tr}\left[(P_cC_c^TR + P_2^T\Lambda B + P_c\Lambda_2^TB)\,\mathrm{d}C_c + (P_2^T\Lambda_2 + P_c\Lambda_c)\right.$$

$$\left.\times\left(\mathrm{d}A_c + \frac{1}{\nu}\,\mathrm{d}B_cR_v^{-1}B_c^T\Lambda_c\right)\right] \tag{55}$$

Setting the coefficients of $\mathrm{d}A_c$, $\mathrm{d}B_c$, and $\mathrm{d}C_c$ to zero, using the trace identities $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, $\operatorname{tr}(A^T) = \operatorname{tr}(A)$, and $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ repeatedly, yields

$$0 = P_2^T\Lambda_2 + P_c\Lambda_c \tag{56}$$

$$\Lambda_2B_c = -\nu C_s^TR_v \tag{57}$$

$$C_c = -R^{-1}B^T(\Lambda P_2 + \Lambda_2P_c)P_c^{-1} \tag{58}$$

Equations (57) and (58) are not explicit for B_c and C_c because the right-hand sides depend on B_c and C_c.

Equations (56–58) with Eqs. (42) and (52) are necessary conditions for the optimal controller for any controller order. However, there is a restriction on the initial compensator state in the differential game (not in application) for the problem to be well-posed. To see this, substitute Eq. (39) into Eq. (38):

$$w_a = -R_a^{-1}\Gamma_a^T\Lambda_ax_a \tag{59}$$

Expanding,

$$v = \frac{1}{\nu}R_v^{-1}B_c^T(\Lambda_2^Tx + \Lambda_cx_c) \tag{60}$$

Substituting Eq. (57),

$$v = -C_sx + \frac{1}{\nu}R_v^{-1}B_c^T\Lambda_cx_c \tag{61}$$

Using this result in Eq. (10),

$$y_s = \frac{1}{\nu}R_v^{-1}B_c^T\Lambda_cx_c \tag{62}$$

Thus, the measurement disturbance feeds back so that the plant information cancels in the measurement! The controller will therefore run open loop, and in general the plant/controller system will be unstable, unless

$$x_c(t) = Tx(t) \tag{63}$$

where T is a transformation matrix.

The particular controller realization is not important, and so without loss of generality we may set $T = I$ for full-order control. Then Eq. (63) yields

$$x_{c_0} = x_0 \tag{64}$$

and a closed-form solution to Eqs. (56–58) with Eqs. (42) and (52) has been found. The well-posedness condition of Eq. (64) is needed *only* in formulating the differential game, not in implementing the resulting controller. In practice it is *not* necessary or expected that the controller initial conditions will be the same as the plant's or in general that Eq. (63) holds because in practice the probability that the disturbances will be worst is zero.

A closed-form solution for other controller orders is not apparent, and further research is needed to investigate the effects of Eq. (63) in these cases. However, the necessary conditions of Eqs. (56–58) with Eqs. (42) and (52) are useful for reduced-order and multiple-plant optimal controller design using gradient techniques.[14,15] Multiple-plant controller optimization is used to increase robustness to plant parameter changes.[16,17]

Solution for Controller Order Equal to Plant Order

In this section a closed-form solution for the full-order LQW controller is derived from the results already obtained for arbitrary order controllers. Some important features of the full-order LQW controller are also derived.

Equations (44) and (63) with the constraint (64) suggest the *assumption* that

$$P_2 = P_c \tag{65}$$

This assumption will be used to find explicit equations for A_c, B_c, and C_c. The assumption will be verified by showing that Eqs. (42), (52), and (56–58) hold under it.

From Eq. (65), since $P_c^T = P_c$,

$$P_2^T = P_2 \tag{66}$$

Therefore, from Eq. (56) and using $\Lambda_c^T = \Lambda_c$,

$$\Lambda_2^T = \Lambda_2 = -\Lambda_c \tag{67}$$

Using Eqs. (65) and (67) in Eq. (58),

$$C_c = -R^{-1}B^TS \tag{68}$$

where

$$S = \Lambda - \Lambda_c \tag{69}$$

To find S, premultiply Eq. (42) by

$$I_\Lambda \triangleq \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \tag{70}$$

and postmultiply by I_Λ^T. All but block (1, 1) yield $0 = 0$. Using Eq. (67), block (1, 1) yields

$$0 = (A^T + C_c^TB^T)(\Lambda - \Lambda_c) + (\Lambda - \Lambda_c)(A + BC_c)$$

$$+ (\Lambda - \Lambda_c)\frac{1}{\nu}\Gamma R_w^{-1}\Gamma^T(\Lambda - \Lambda_c) + C^TQC + C_c^TRC_c \tag{71}$$

Substituting Eqs. (68) and (69) and rearranging,

$$0 = A^TS + SA - S\left(BR^{-1}B^T - \frac{1}{\nu}\Gamma R_w^{-1}\Gamma^T\right)S + C^TQC \tag{72}$$

a Riccati equation for S in terms of *given* information. Thus, once Eq. (72) is solved, C_c can be found from Eq. (68).

To find B_c, expand Eq. (42) using Eq. (67). The (1, 1) block is

$$0 = A^T\Lambda - C_s^TB_c^T\Lambda_c + \Lambda A - \Lambda_cB_cC_s + \frac{1}{\nu}\Lambda_c\Gamma R_w^{-1}\Gamma^T\Lambda$$

$$+ \frac{1}{\nu}\Lambda_cB_cR_v^{-1}B_c^T\Lambda_c + C^TQC \tag{73}$$

the (1, 2) block is

$$0 = (-A^T + C_s^TB_c^T)\Lambda_c + \Lambda BC_c - \Lambda_cA_c$$

$$-\frac{1}{\nu}\Lambda\Gamma R_w^{-1}\Gamma^T\Lambda_c - \frac{1}{\nu}\Lambda_cB_cR_v^{-1}B_c^T\Lambda_c \tag{74}$$

and the (2,2) block is

$$0 = (-C_c^TB^T + A_c^T)\Lambda_c + \Lambda_c(-BC_c + A_c) + \frac{1}{\nu}\Lambda_c(\Gamma R_w^{-1}\Gamma^T + B_cR_v^{-1}B_c^T)\Lambda_c + C_c^TRC_c \quad (75)$$

Using Eqs. (57) and (67) in the (1, 1) block yields

$$0 = A(\nu\Lambda^{-1}) + (\nu\Lambda^{-1})A^T - (\nu\Lambda^{-1})(C_s^TR_vC_s - \frac{1}{\nu}C^TQC) \times (\nu\Lambda^{-1}) + \Gamma R_w^{-1}\Gamma^T \quad (76)$$

a Riccati equation for $\nu\Lambda^{-1}$ in terms of *given* information. Then from Eqs. (57), (67), and (69),

$$B_c = \nu(\Lambda - S)^{-1}C_s^TR_v = \nu\left[\Lambda(I - \Lambda^{-1}S)\right]^{-1}C_s^TR_v \quad (77)$$

so

$$B_c = \left[I - \frac{1}{\nu}(\nu\Lambda^{-1})S\right]^{-1}(\nu\Lambda^{-1})C_s^TR_v \quad (78)$$

an explicit equation for B_c.

To find A_c, sum Eqs. (74) and (75) and transpose:

$$0 = \Lambda_c(-A + B_cC_s - BC_c + A_c) + C_c^TB^T(\Lambda - \Lambda_c) - \frac{1}{\nu}\Lambda_c\Gamma R_w^{-1}\Gamma^T(\Lambda - \Lambda_c) + C_c^TRC_c \quad (79)$$

Substituting Eqs. (69) and (68) for all but the first appearance of C_c in Eq. (79) and canceling terms,

$$A_c = A + \frac{1}{\nu}\Gamma R_w^{-1}\Gamma^TS + BC_c - B_cC_s \quad (80)$$

Now we shall verify the assumption of Eq. (65). To do this, note Eqs. (42), (52), and (56–58) are nine independent equations in the nine unknowns Λ, Λ_2, Λ_c, P, P_2, P_c, A_c, B_c, and C_c [since Eqs. (42) and (52) are symmetric, they each contain three independent block equations]. Three independent combinations of the three independent block equations in Eq. (42) have been used to obtain Eqs. (72), (76), and (80) so that the solution is consistent with Eq. (42). Equations (56–58) have been used repeatedly in developing the solution so that the solution is consistent with them. It remains to check Eq. (52). Since no assumption has been made concerning P, P follows from block (1, 1) of Eq. (52), and the solution is consistent with this. Using Eq. (65) and $x_c(0) = x_0$ in blocks (1, 2) and (2, 2) of Eq. (52) yields the same equation for P_c from each block so that the solution is consistent with Eq. (52). Therefore, the assumption of Eq. (65) is verified. Furthermore, the three block equations from Eq. (52) are exactly the same if

$$P = P_2 = P_c \quad (81)$$

thereby proving Eq. (81).

Equations (68), (78), and (80) with (72) and (76) give the controller that solves Eqs. (5–14), which is the same as the H_∞ controller in Refs. 3 and 4 for $\nu = \gamma^2$. It remains to find the worst disturbances and initial conditions and the saddle-point cost.

Partitioning Eq. (59) using Eq. (67),

$$w = \frac{1}{\nu}R_w^{-1}\Gamma^T[\Lambda \quad -\Lambda_c]x_a \quad (82)$$

$$v = \frac{1}{\nu}R_v^{-1}B_c^T[-\Lambda_c \quad \Lambda_c]x_a \quad (83)$$

Since $x_c(0) = x_0$, from Eq. (63) *assume*

$$x_c(t) = x(t) \quad (84)$$

Now check this assumption. Under it, Eq. (82) with Eq. (69) yields

$$w = K_wx \quad (85)$$

where

$$K_w = \frac{1}{\nu}R_w^{-1}\Gamma^TS \quad (86)$$

and Eq. (83) yields

$$v(t) = 0 \quad (87)$$

Substituting Eq. (85) with Eq. (86) into Eq. (8),

$$\dot{x} = Ax + Bu + \frac{1}{\nu}\Gamma R_w^{-1}\Gamma^TSx, \qquad x(0) = x_0 \quad (88)$$

Substituting Eq. (87) into Eq. (10) and then this result and Eq. (80) into Eq. (11),

$$\dot{x}_c = \left(A + \frac{1}{\nu}\Gamma R_w^{-1}\Gamma^TS + BC_c - B_cC_s\right)x_c + B_cC_sx$$

$$x_c(0) = x_0 \quad (89)$$

Canceling terms and using Eq. (12),

$$\dot{x}_c = Ax_c + Bu + \frac{1}{\nu}\Gamma R_w^{-1}\Gamma^TSx_c, \qquad x_c(0) = x_0 \quad (90)$$

Comparing Eq. (88) with Eq. (90), the same differential equation governs x and x_c, with the same initial condition for x and x_c. Therefore, the assumption of Eq. (84) is verified.

The best controls/worst disturbances solution is balanced on a pinhead: if $x_{c_0} \neq x_0$ or some linear combination of x_0 as in Eq. (63), the disturbances will become infinite as the system goes unstable and the problem is ill posed; if $x_{c_0} = x_0$, the worst measurement disturbance is identically zero and the worst process disturbance is a feedback on the plant states, which from Eqs. (88) and (90) are the same as the controller states in the differential game. An interpretation is that the controller must know the initial plant state for a fair game since the disturber does, since it feeds back on it.

With $\nu = 0$, substituting Eq. (85) into Eq. (14),

$$W = \int_0^\infty x^TK_w^TR_wK_wx \, dt \quad (91)$$

Using the trace identity and the (1, 1) block of Eq. (44),

$$W = \mathrm{tr}(PK_w^TR_wK_w) \quad (92)$$

As with many optimization problems, W is determined after the fact since K_w depends on the Lagrange multiplier ν, and there is no apparent closed-form solution for ν in terms of W, and so ν must be prescribed. As ν decreases, W increases. A desired W can be obtained by performing a bisection algorithm on ν. However, there is no solution below a critical value of $\nu = \gamma^2$ where W and J_S both approach infinity. The solution at the critical value is called the "optimal" H_∞ controller in Refs. 3 and 4. We call it the infinite-disturbance LQW controller. Above the critical value of ν, the finite-disturbance LQW ("suboptimal" H_∞) controller results. When $\nu \to \infty$, it is apparent from Eqs. (72), (76), (68), (78), and (80) that the controller becomes the classic LQG controller if R_w^{-1} is identified with the process noise spectral density, R_v^{-1} with the measurement noise spectral density, and $\nu\Lambda^{-1}$ with the estimate-error covariance matrix. This is noted in Ref. 4. References 3 and 4 give five conditions involving S and $\nu\Lambda^{-1}$ and their associated Riccati equations (72) and (76) that must be satisfied for a value of $\nu=\gamma^2$ to be greater than critical. In practice, these conditions can be checked, or alternately ν is above critical if a solution is found that stabilizes the closed-loop system with the disturbance feedback.

It remains to find the worst initial conditions and the saddle-point cost. From Eq. (43), since we have satisfied $\mathrm{d}Z_a=0$,

$$\mathrm{d}\bar{J} = x_{a_0}^T(\Lambda_a - \eta I)\,\mathrm{d}x_{a_0} \tag{93}$$

Now since $x_{c_0}=x_0$ from Eq. (64),

$$\mathrm{d}x_{c_0} = \mathrm{d}x_0 \tag{94}$$

and so using this, Eq. (67), and Eq. (69) into Eq. (93),

$$\mathrm{d}\bar{J} = x_0^T(S - 2\eta I)\,\mathrm{d}x_0 \tag{95}$$

Therefore, 2η must be an eigenvalue of S with x_0 the corresponding eigenvector, but it is not yet apparent which eigenpair maximizes J.

To find the saddle-point cost, from Eq. (27) with Eq. (44),

$$J = \tfrac{1}{2}\mathrm{tr}(P_a Q_a) \tag{96}$$

Substituting for Q_a from Eq. (42),

$$J = \tfrac{1}{2}\mathrm{tr}\left[-P_a(A_a^T\Lambda_a + \Lambda_a A_a - \Lambda_a\Gamma_a R_a^{-1}\Gamma_a^T\Lambda_a)\right] \tag{97}$$

Using the trace identities $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ and $\mathrm{tr}(A+B)=\mathrm{tr}(A)+\mathrm{tr}(B)$,

$$\begin{aligned} J = \tfrac{1}{2}\mathrm{tr}\Big\{\big[-P_a(A_a^T - \Lambda_a\Gamma_a R_a^{-1}\Gamma_a^T) \\ -(A_a - \Gamma_a R_a^{-1}\Gamma_a^T\Lambda_a)P_a\big]\Lambda_a\Big\} \\ -\tfrac{1}{2}\mathrm{tr}(\Gamma_a R_a^{-1}\Gamma_a^T\Lambda_a P_a\Lambda_a) \end{aligned} \tag{98}$$

which, using Eq. (52) with Eq. (50) and trace identities, becomes

$$J = \tfrac{1}{2}\mathrm{tr}(x_{a_0}x_{a_0}^T\Lambda_a) - \tfrac{1}{2}\mathrm{tr}(P_a\Lambda_a\Gamma_a R_a^{-1}\Gamma_a^T\Lambda_a) \tag{99}$$

Using $x_c(0)=x_0$ and Eq. (67) in the first term and performing the block multiplications in the second term using Eqs. (67) and (81),

$$\begin{aligned} J = \tfrac{1}{2}\mathrm{tr}\left[x_0x_0^T(\Lambda-\Lambda_c)\right] - \tfrac{1}{2}\mathrm{tr}\Big[P(\Lambda-\Lambda_c) \\ \times\left(-\frac{1}{\nu}\Gamma R_w^{-1}\Gamma^T\right)(\Lambda-\Lambda_c)\Big] \end{aligned} \tag{100}$$

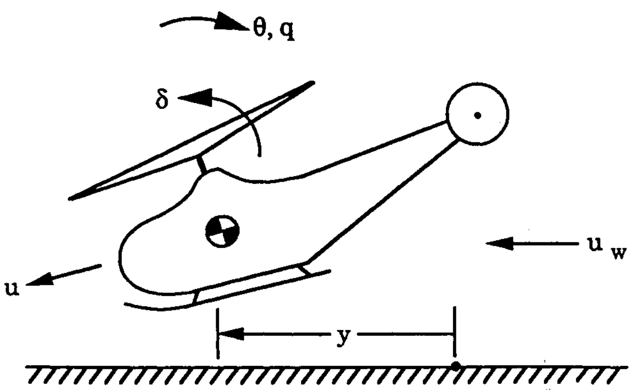


Fig. 1 Helicopter near hover.

which, using Eqs. (69) and (85) with Eq. (86), becomes

$$J = \tfrac{1}{2}\mathrm{tr}(x_0x_0^TS) + \tfrac{1}{2}\mathrm{tr}(P\nu K_w^T R_w K_w) \tag{101}$$

Finally, using the trace identity and substituting Eq. (92),

$$J = \tfrac{1}{2}(x_0^TSx_0 + \nu W) \tag{102}$$

Since W depends on x_0, it is not apparent that 2η necessarily is the maximum eigenvalue of S. Thus,

$$\begin{aligned} 2\eta &= \text{eigenvalue of } S \text{ that maximizes Eq. (102)} \\ x_0 &= \text{corresponding unit eigenvector} \end{aligned} \tag{103}$$

In summary, Eqs. (72), (76), (68), (78), and (80) give the linear-quadratic-best-control/worst-bounded-disturbances (LQW) or so-called "H_∞" full-order controller. The worst process disturbance is a feedback on the plant states, the worst measurement disturbance is zero (!), and Eq. (103) gives the worst initial condition.

Comparison of LQG Control to LQW (H_∞) Control

The purpose of the example in this section is to illustrate features of LQW control and differences between LQW and LQG control that are of practical importance. Although this example does not, of course, constitute a proof of the general relative merits of LQG and LQW control, we believe the qualitative statements made apply in general based on experience and physical reasoning.

Figure 1 shows a helicopter near hover disturbed by horizontal wind gusts. The plant model is[19]

$$\begin{bmatrix}\dot{u}\\ \dot{q}\\ \dot{\theta}\\ \dot{y}\end{bmatrix} = \begin{bmatrix} X_u & X_q & -g & 0\\ M_u & M_q & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\ q\\ \theta\\ y\end{bmatrix} + \begin{bmatrix}X_\delta\\ M_\delta\\ 0\\ 0\end{bmatrix}\delta + \begin{bmatrix}-X_u\\ -M_u\\ 0\\ 0\end{bmatrix}u_w \tag{104}$$

where g is the gravitational force per unit mass, u is the forward velocity, q is the pitch angular velocity, θ is the pitch angle, y is the position deviation from desired hover point, δ is the longitudinal cyclic stick deflection, and u_w is the horizontal wind velocity. There is a single measurement, of position (y).

We take as the performance index

$$J = E\left[\lim_{t_f\to\infty}\frac{1}{2t_f}\int_0^{t_f}(y^2+\delta^2)\,\mathrm{d}t\right] \tag{105}$$

where y is in feet and δ is in deci-inches. For LQG controller design, the process disturbance u_w and the additive measurement disturbance are assumed to be white noise with spectral densities of 18 ft^2/s and 0.4 ft^2 s, respectively.

For an OH-6A helicopter,[20] the nominal values of the six plant parameters are

$$\begin{aligned} &(X_u, X_q, M_u, M_q, X_\delta, M_\delta) \\ &\quad = (-0.0257, 0.013, 1.26, -1.765, 0.086, -7.408) \end{aligned} \tag{106}$$

where the units are feet, seconds, and centi-radians (crad), and δ is in deci-inches.

The full-order LQW controller can be found by solving Eqs. (72), (76), (68), (78), and (80) using standard software packages such as MATLAB[21] or by using the MATLAB μ-Analysis and Synthesis Toolbox.[8] If a Riccati solver is not available, one is made easily by modifying the LQR or LQG subroutine to solve general Riccati equations.

Table 1 OH-6A helicopter poles and zeros

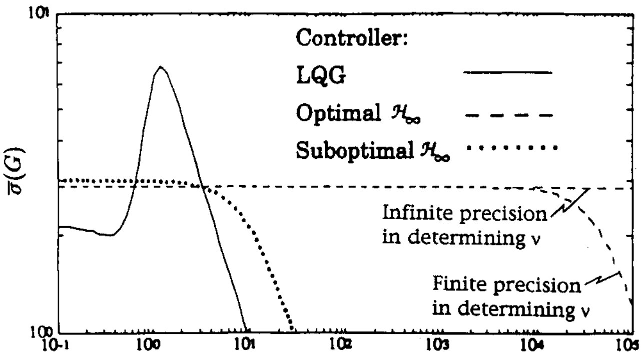
Poles, all values in rad/s	Zeros, all values in rad/s
0	$-0.32 \pm 5.3j$
-1.9	
$0.049 \pm 0.46j$	

Table 2 Controller poles and zeros for OH-6A helicopter

	Controller, all values in rad/s		
	LQG	Inf.-dist. LQW (optimal H_∞)	Finite-dist. LQW (suboptimal H_∞)
Poles	$-0.47 \pm 1.9j$	$-0.92 \pm 1.7j$	$-0.85 \pm 1.7j$
	$-2.5 \pm 1.2j$	-2.8	-3.0
		$-4.7E4$	-9.7
Zeros	$-0.14 \pm 0.37j$	$-0.083 \pm 0.33j$	-0.079 ± 0.33
	-1.9	-1.9	-1.9
Plant/controller open-loop bandwidth	0.94	0.85	0.84

Table 3 OH-6A helicopter rms responses to white-noise disturbances

	Controller		
	LQG	Inf.-dist. LQW (optimal H_∞)	Finite-dist. LQW (suboptimal H_∞)
y, ft	4.5	5.5	5.7
δ, deci-in.	3.4	280	4.1



ω (*rad/sec*)

Fig. 2 Maximum singular value of closed-loop transfer function vs frequency for OH-6A: LQG/LQW (H_∞) controller comparison.

The infinite-disturbance LQW (optimal H_∞) controller for the OH-6A was found using a bisection on $\nu = \gamma^2$. The critical value of ν is 8.2592. A finite-disturbance (suboptimal H_∞) controller then was found for $\nu = 9$.

Table 1 lists the plant poles and zeros. Table 2 lists the poles and zeros for the LQG, infinite-disturbance LQW, and finite-disturbance LQW controller. The poles and zeros are not plotted because the most interesting differences are off scale, and otherwise the two LQW controllers would be difficult to tell apart. Each controller has a real-axis zero at approximately the same point, -1.9 rad/s. Each controller notches the unstable plant pole, but the LQW notches are detuned relative to the LQG notch. The largest differences are in the other two poles, which move to the real axis with the LQW controllers. In particular, as ν decreases toward its critical value, a real-axis pole moves toward $-\infty$. The value shown for the infinite-disturbance LQW controller is limited only by the precision with which ν is specified. Effects of this pole will be discussed later.

Table 2 also shows the bandwidth (measurement y_s to measurement magnitude crossover) with each controller. The LQG bandwidth is 11% higher than the infinite-disturbance LQW bandwidth and 13% higher than the finite-disturbance LQW. However, in terms of ν the finite-disturbance LQW controller lies between the infinite-disturbance LQW controller and the LQG controller, for which $\nu \to \infty$. Therefore, the only general comparison to be made about bandwidth is that it does not differ greatly from LQG to LQW.

Figure 2 shows the maximum singular value of the closed-loop transfer matrix $G(j\omega)$ [defined in Eq. (2)] vs frequency for the OH-6A with each controller. The LQG controller has a peak over a small frequency band within which intelligent disturbances could degrade performance, but the system is less susceptible to disturbances at other frequencies than with the LQW controllers. The finite-disturbance LQW controller exhibits the same high-frequency gain as the LQG controller. However, the infinite-disturbance LQW controller has a completely flat closed-loop frequency response (the slight drop for the highest frequencies shown is due only to the finite precision of ν). This is the result of a real-axis controller pole that approaches $-\infty$. As noted in Ref. 9, with the infinite-disturbance LQW controller the controller and disturber are at a standoff. The controller flattens the frequency response so that there is no frequency at which the disturber can gain an advantage. This is unrealistic. The infinite closed-loop bandwidth makes the controller work against high-frequency disturbances that otherwise would not excite the system.[9] This results in infinite control energy. Of course, in practice the controller bandwidth is limited by actuator dynamics, but the infinite-disturbance LQW controller would still push the actuator limits and is probably always impractical.

This is the reason that proponents of H_∞ control do not use "straight" (i.e., without μ synthesis) optimal H_∞ control but instead reduce the closed-loop bandwidth using μ synthesis[4,8] with *high*-pass filters at the disturbance-injection points.[6-8] The benefits of this approach over LQG control and parameter-robust LQG control[16,17] are not clear, especially since μ synthesis yields higher-order controllers than LQG design and is restricted to square systems (same number of inputs and outputs for closed-loop system).

The rms response to process and measurement white noise disturbances with each controller is shown in Table 3. This comparison is made to indicate performance in the presence of high-frequency disturbances, realizing that the LQG controller is optimized for white-noise disturbances. The performance of the infinite-disturbance LQW controller is infinitely bad for an infinitely precise ν.

Infinite-disturbance LQW (optimal H_∞) control appears to be impractical. Finite-disturbance LQW (suboptimal H_∞) control is practical, depending on the degree of subcriticality as specified by ν, but is quite similar to LQG control, which is the limiting case of infinite-disturbance LQW control. LQW control is more conservative than LQG with regard to *worst* bounded, deterministic disturbances. However, LQW control is less conservative for higher-frequency disturbances and, at least for this example, for all disturbances outside a small frequency range in which the closed-loop LQG frequency response peaks.

Conclusions

A new calculus-of-variations H_∞ compensator derivation shows that the H_∞ problem is a differential game between the controls and the disturbances. The H_∞ controller is the *linear-quadratic* best controller for the *worst* bounded disturbances, or *LQW* controller. For the full-order LQW controller, the worst process disturbance is a feedback on the plant states, the worst measurement disturbance is zero (!), and the controller initial conditions must be set equal to the plant initial conditions for a well-posed differential game. Necessary conditions have been found for reduced-order LQW controllers and higher-order LQW controllers, useful in multiple-plant optimization for robustness. Comparisons of LQG to LQW control for a helicopter near hover show that infinite-disturbance LQW (optimal H_∞) control is not practical because it is sensitive to high-frequency disturbances. Although μ-synthesis design using *high*-pass filters on disturbances decreases this sensitivity, the advantages over LQG techniques are not clear since controller order is higher, design is restricted to square systems, and techniques for designing LQG-based controllers for real parameter robustness now exist. Finite-disturbance LQW (suboptimal H_∞) control has characteristics between infinite-disturbance LQW and LQG control, which is the limiting case of finite-disturbance LQW control.

References

[1]Zames, G., "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses," *IEEE Transactions on Automatic Control,* Vol. AC-26, 1981, pp. 301–320.

[2]Doyle, J. C., "Analysis of Feedback Systems with Structured Uncertainty," *IEEE Proceedings,* Vol. 129, Pt. D, No. 6, 1982, pp. 242–250.

[3]Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Transactions on Automatic Control,* Vol. AC-34, No. 8, 1989, pp. 831–847.

[4]Dailey, R. L., "Lecture Notes for the Workshop on H_∞ and μ Methods for Robust Control," 1990 American Control Conf., San Diego, CA, May 1990.

[5]Yeh, H.-H., Banda, S. S., Heise, S. A., and Bartlett, A. C., "Robust Control Design with Real-Parameter Uncertainty and Unmodeled Dynamics," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 6, 1990, pp. 1117–1125.

[6]Doyle, J. C., Lenz, K., and Packard, A., "Design Examples using μ-Synthesis: Space Shuttle Lateral Axis FCS During Reentry," *Modelling, Robustness and Sensitivity Reduction in Control Systems,* edited by R. F. Curtain, Vol. F34, NATO ASI Series, Springer-Verlag, Berlin, 1987, pp. 127–154.

[7]Balas, G. J., Chu, C.-C., and Doyle, J. C., "Vibration Damping and Robust Control of the JPL/AFAL Experiment Using μ-Synthesis," *Proceedings of the 28th Conference on Decision and Control* (Tampa, FL), IEEE, Piscataway, NJ, Dec. 1989, pp. 2689–2694.

[8]Balas, G. J., Doyle, J. C., Glover, K., Packard, A., and Smith, R., *μ-Analysis and Synthesis Toolbox User's Guide,* The MathWorks, Inc., South Natick, MA, and MUSYN, Inc., Minneapolis, MN, April 1991.

[9]Bryson, A. E., and Carrier, A., "A Comparison of Control Synthesis Using Differential Games (H_∞) and LQR," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Boston, MA), AIAA, Washington, DC, Aug. 1989, pp. 1422–1427.

[10]El Ghaoui, L., Carrier, A., and Bryson, A. E., "Linear-Quadratic Minimax Controllers," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 4, 1992, pp. 953–961.

[11]Mills, R. A., and Bryson, A. E., "Minimax State Feedback Design with Norms on Initial Conditions, Disturbances, and Parameter Changes," *Proceedings of the 1991 American Control Conference* (Boston, MA), American Automatic Control Council, Evanston, IL, 1991, pp. 1917, 1918.

[12]Bernhard, P., "A Min-Max Certainty Equivalence Principle, and Its Application to Continuous Time, Sampled Data, and Discrete Time H_∞ Optimal Control," *Rapports de Recherche,* Programme 5, No. 1347, Inst. National de Recherche en Informatique et en Automatique (INRIA), Sophia Antipolis, France, Dec. 1990.

[13]Rhee, I., and Speyer, J. L., "A Game Theoretic Approach to a Finite-Time Disturbance Attenuation Problem," *IEEE Transactions on Automatic Control,* Vol. AC-36, No. 9, Sept. 1991, pp. 1021–1032.

[14]Ly, U.-L., "A Design Algorithm for Robust Low-Order Controllers," Ph.D. Thesis, Stanford Univ., Dept. of Aeronautics and Astronautics, Rept. 536, Stanford, CA, Nov. 1982.

[15]Ly, U.-L., Cannon, R. H., and Bryson, A. E., "Design of Low-Order Compensators Using Parameter Optimization," *Automatica,* Vol. 21, No. 3, 1985, pp. 315–318.

[16]Mills, R. A., and Bryson, A. E., "Parameter-Robust Control Design Using a Minimax Method," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 5, 1992, pp. 1068–1075.

[17]Mills, R. A., "Robust Controller and Estimator Design using Minimax Methods," Ph.D. Thesis, Stanford Univ., Dept. of Aeronautics and Astronautics, Rept. 618, Stanford, CA, March 1992.

[18]Boyd, S. P., and Barratt, C. H., *Linear Controller Design: Limits of Performance,* Prentice-Hall, Englewood Cliffs, NJ, 1991, Chap. 5.

[19]Bryson, A. E., *Control of Spacecraft and Aircraft,* Princeton Univ. Press, Princeton, NJ, Chap. 14 (in preparation).

[20]Heffley, R. K., Jewell, W. F., Lehman, J. M., and Van Winkle, R. A., "A Compilation and Analysis of Helicopter Handling Qualities Data," NASA CR 3144, Aug. 1979.

[21]Anon., *PRO-MATLAB User's Guide,* The MathWorks, Inc., South Natick, MA, Jan. 1990.